# Errata

**Document Title:** Digital Signal Analysis: Time and
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## HP References in this Application Note

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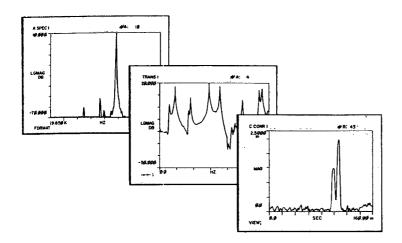
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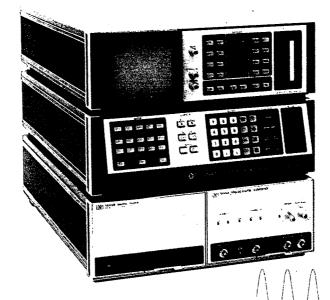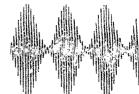
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# APPLICATION NOTE

**240-0**

# INTRODUCTION

In the field of digital signal analysis many measurement techniques are available. Each measurement has its advantages and disadvantages, depending on the application and the nature of the signal or signals being measured.

The purpose of this note is to provide a primary understanding of what each measurement is, what information it provides, and some examples of where it is used. It is not the intent of this note to provide detailed mathematical derivations or theory. This information is available in textbooks and other material, some of which are listed at the end of this note.

Often, more than one measurement can be used to obtain the same or similar results. It is not always clear which will provide the maximum information. There are no simple solutions to this dilema—only the continued use of each measurement and an understanding of the dynamics of the particular application will provide the necessary expertise.

This situation does point out the importance of high performance and flexibility of measurements that are required of any digital signal analyzer.
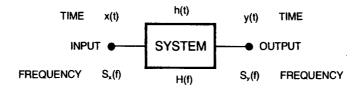
# CONTENTS

# GLOSSARY OF TERMS

The terms and relationships used throughout this note are defined below.

**Linear Relationships**

TIME     x(t)      h(t)      y(t)    TIME

INPUT ●——[ SYSTEM ]——● OUTPUT

FREQUENCY   $S_x(f)$     H(f)     $S_y(f)$   FREQUENCY

where:    x(t)    = Time domain input to the system
           y(t)    = Time domain output of the system
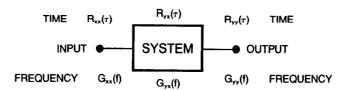           $S_x(f)$   = Linear Fourier spectrum of x(t)
           $S_y(f)$   = Linear Fourier spectrum of y(t)
           H(f)    = System transfer function (frequency response)
           h(t)    = System impulse response

**Square Law Relationships**

TIME    $R_{xx}(\tau)$     $R_{yx}(\tau)$     $R_{yy}(\tau)$    TIME

INPUT ●——[ SYSTEM ]——● OUTPUT

FREQUENCY   $G_{xx}(f)$     $G_{yx}(f)$     $G_{yy}(f)$   FREQUENCY

where:    $R_{xx}(\tau)$ = Auto correlation of the input signal x(t)
           $R_{yy}(\tau)$ = Auto correlation of the output signal y(t)
           $G_{xx}(f)$ = Auto power spectrum of x(t)
           $G_{yy}(f)$ = Auto power spectrum of y(t)
           $G_{yx}(f)$ = Cross power spectrum of y(t) and x(t)
           $R_{yx}(f)$ = Cross correlation of y(t) and x(t)

**Derived Relationships[1]**

$$H(f) = \frac{G_{yx}(f)}{G_{xx}(f)} = \frac{S_y(f)}{S_x(f)} \cdot \frac{S_x^*(f)}{S_x^*(f)}$$

$$\gamma^2(f) = \frac{\overline{|G_{yx}(f)|^2}}{\overline{G_{xx}(f)}\ \overline{G_{yy}(f)}}$$

where: $\gamma^2(f)$ = Coherence function of the measurement

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[1]The star (*) denotes the conjugate of the function; the bar (—) denotes the average function.

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## TIME RECORD AVERAGING

The time record is the linear representation of a signal in the time domain. For pure periodic signals one record is sufficient to describe the signal. However, if the signal contains noise or other non-related signals, then averaging is usually required to 'clean up' the signal. A trigger condition, related to the signal of interest, is necessary. This can be provided by an external signal or internally by the waveform itself. Signals which are synchronous with the trigger will average to their mean value while noise or non-synchronous signals will average to zero. For this reason, time record averaging is frequently called 'synchronous time averaging'.

Time record averaging is useful in extracting a signal or signals from noise of approximately the same frequency content. For example, sonar pulses hidden in the random noise of the ocean, or the ECG waveform of a patient's heart hidden in the noise of random muscle and nerve reactions.
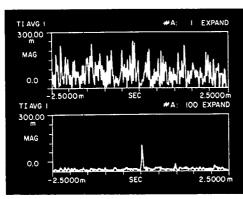


Figure 1

**Fig. 1** *Upper trace shows an impulse buried in noise. Without time record averaging the signal is indiscernible from the noise. The bottom trace is the same signal after stable averaging 100 records. Note the average noise level is approaching zero, since it is not synched to the trigger, making the synched signal clearly visible. In general, the improvement in signal-to-noise ratio is proportional to the square root of the number of independent averages. In dB, the equation is : S/N (dB) = 20 log (N)$^{1/2}$.*

1

## AUTO CORRELATION

The correlation function $R_{xx}(\tau)$ is a special time average defined by

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_T x(t)\, x(t+\tau)\, dt$$

Thus, the correlation function is found by taking a signal, multiplying it by the same signal displaced $\tau$ units in time and averaging the product over all time.

For the sake of simplicity and speed most digital signal analyzers perform the correlation operation by taking advantage of its duality with the power spectrum; that is,

$$R_{xx}(\tau) \longleftrightarrow G_{xx}(f)$$

Thus,

$$R_{xx}(\tau) = F^{-1}\left[G_{xx}(f)\right] = F^{-1}\left[S_x(f) \cdot S_x(f)^*\right]$$

The auto correlation function always has a maximum at $\tau=0$ equal to the mean square value of $x(t)$. If the signal $x(t)$ is periodic, the correlation function is also periodic with the same period. Random noise, on the other hand, will only correlate around $\tau=0$.

The auto correlation function can be used to improve the signal-to-noise ratio of periodic signals. The random noise component will concentrate near $\tau=0$ while the periodic component will repeat itself with the same periodicity as the signal.

Also, impulsive type signals such as pulse trains, bearing ping, or gear chatter will show up more distinctly in correlation or time record averaging than in a frequency domain analysis.
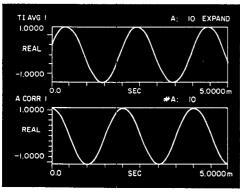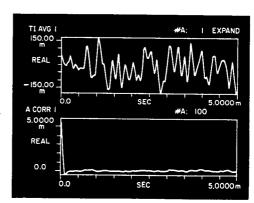


Figure 2A



Figure 2B
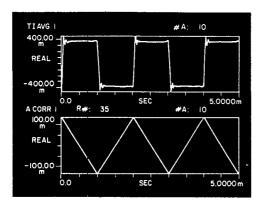
Figure 2C

**Fig. 2A, 2B, & 2C** *These figures show the signal and its auto correlation function for a sinusoid, random noise, and a square wave respectively. Note that the τ=0 value of each auto correlation function is the mean square value of the signal.*
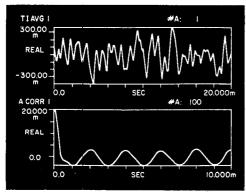


Figure 3

**Fig. 3** *The upper trace shows the sum of random noise and a sinewave. The sinewave is not noticeable. The lower trace shows the auto correlation of the signal. The auto correlation of the noise is the peak centered at τ=0. The auto correlation of the sinusoid is clearly visible and has the same periodicity as the signal itself.*

## CROSS CORRELATION

Cross correlation is a measure of the similarity between two signals.

$$R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_T y(t)\, x(t+\tau)\, dt$$

It is calculated the same way as the auto correlation using the dual relationship

$$R_{yx}(\tau) \longleftrightarrow G_{yx}(f)$$

Thus,

$$R_{yx}(\tau) = F^{-1}[S_y(f) \cdot S_x(f)^*]$$

Cross correlation indicates the similarity between two signals as a function of time shift ($\tau$). The most useful application for the cross correlation function is to determine time delays between signals.

These signals can be impulsive signals such as encountered in radar and sonar applications, or broadband random noise such as encountered in system stimulus response measurements. Examples of the latter are the measurement of transmission path delays, room acoustics, airborne noise analysis, and noise source identification.
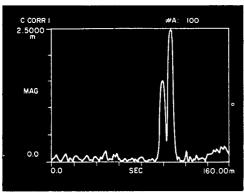


Fig. 4 Cross correlation can be used to determine noise transmission paths. Unwanted noise and vibrations often travel via different paths. Correlation can be used to isolate these paths. Figure 4 displays two peaks with time delays representative of the individual transmission path delays in an electronic circuit. Examination of the amplitude of the peaks will give a measure of the relative amounts of noise transmitted by each path.

Figure 4

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## IMPULSE RESPONSE

The impulse response is the time domain representation of the transfer characteristics of a system. It is defined in the 5420A as:

$$h(t)=F^{-1}\,[H(f)]$$

The impulse response is frequently a more sensitive indicator of time delays between signals than the cross correlation function. Mathematically,

$$R_{yx}(\tau)=h(t)*R_{xx}(\tau)$$

which says that $R_{yx}(\tau)$ is a convolution of h(t) with the auto correlation $R_{xx}(\tau)$. The measurement of interest represented by h(t) is confused by the irrelevant information contained in $R_{xx}(\tau)$.

The impulse response can be used to characterize electronic filters, measure acoustic absorption coefficients, determine noise transmission paths, and identify structural modes of vibration.
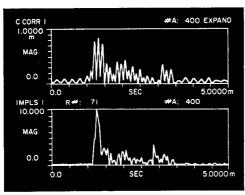


Figure 5

**Fig. 5** *Measurement results of a speaker/microphone system where the microphone was in a normal room environment approximately one foot from the speaker. Note how the cross correlation lacks the definition of the impulse response in this case, due to the dispersive nature of the signals (i.e., the smearing effect of the auto correlation function $R_{xx}(\tau)$). The cross correlation shown in Figure 4, on the other hand, is due to pure time delays in an electrical circuit with no other interfering signals. The smearing effect is especially evident in the second delay in Figure 5 at approximately 3 ms, which is quite clear in the impulse response but completely masked in the cross correlation. This second delay was due to a reflection of sound off a small, hard surface approximately one foot from the microphone.*

## AMPLITUDE HISTOGRAM

The histogram is a representation of the probability density of the input waveform. The abscissa of the display is calibrated so that the left most point represents the negative of the full-scale voltage setting and the right most point the positive value of the full-scale setting. The center of the display then is zero volts. The 5420A samples the input waveform, computes the voltage of the sample, and adds one to the channel corresponding to that voltage on the display. This continues until the selected number of averages have been reached. Dividing the histogram by 512 times the number of averages normalizes it to the total number of samples.
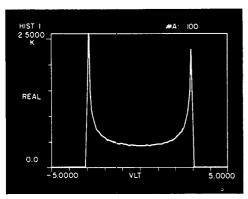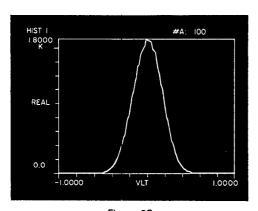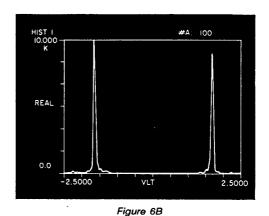
Figure 6A

Figure 6B

Figure 6C

**Fig. 6A, 6B & 6C** *Typical histograms for sinusoids, square waves, and random noise respectively.*

## LINEAR SPECTRUM AVERAGING

The linear spectrum is the Fourier transform of the time signal x(t). Thus,

$$S_x(f) = F[x(t)]$$

The linear spectrum gives both magnitude and absolute phase information at each frequency in the analysis band. As such, it requires a trigger condition for averaging. As with time record averaging, any non-synchronous signals will average to zero. This gives the linear transform extremely good dynamic range characteristics.

Linear spectrum averaging is useful for extracting fine-line components due to rotating members of a jet engine from the random noise of combustion, or extracting known radio frequencies from the background noise of space.

*Often in rotating machinery analysis, the signals of interest are buried in background noise. In addition, if the machine's speed is varying, the frequencies of the signals will vary. If the sampling rate of the analyzer can be locked to the machine's speed, the variation in frequency can be eliminated and the signals can be averaged.*
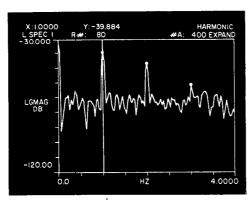


Figure 7

**Fig. 7** *A linear spectrum was measured using this technique along with the necessary trigger signal. Each major peak represents an order of rotation (i.e., multiple of the shaft speed) and instead of frequency the horizontal axis is scaled in terms of orders of rotation. The peak at 1 is the fundamental shaft speed and the intensified dots are its harmonics.*
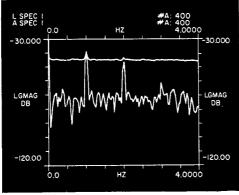


Figure 8

**Fig. 8** *When a trigger signal is available and the noise level is high, linear spectrum averaging can provide a better measurement than auto spectrum averaging, which is described next. This is evidenced in Figure 8 where the top trace is the auto spectrum and the lower trace the linear spectrum of the same signal. Note that the peaks are equivalent on a dB basis but that the noise is averaging to its mean square value with the auto spectrum and its mean value (zero) with the linear spectrum.*

## AUTO POWER SPECTRUM

The auto spectrum is the magnitude squared of the linear spectrum; that is,

$$G_{xx}(f) = S_x(f) \cdot S_x(f)^*$$

$G_{xx}(f)$ then is a real valued function containing magnitude information only (no phase) and is independent of any trigger point. It can be averaged without the need of a synchronizing trigger. The auto spectrum is also the Fourier transform of the auto correlation function and contains the same information in a different format.

The auto power spectrum can be represented in three ways. As a power spectrum it has the units of volts squared and is useful for analyzing spectra with discrete spectral components. However, with broadband signals, such as random noise or transients, the absolute level will vary with the analysis bandwidth. To overcome this, the power spectrum is normalized to a 1 Hz bandwidth which makes measurements made with different analysis ranges comparable one to another. The normalized auto power spectrum is referred to as a power spectral density or PSD measurement. With transient data, energy is a more useful concept than power (since the average power in a transient signal is theoretically zero) and the spectrum is referred to as an energy spectral density or ESD measurement. An ESD measurement has the units $(V^2\text{–}s)/Hz$.

The auto spectrum can be used to look for periodic signals in noisy environments such as might be found in aircraft vibrations, rotating machinery, and automotive noise. It can also be used to characterize nonlinear distortion in electronic circuits and the phase noise of precision frequency sources, for example.

*If a trigger signal is available, linear spectrum averaging is a useful technique. Otherwise, power spectrum averaging can be used. The signal-to-noise ratio of a power spectrum measurement can be improved using band selectable analysis (BSA). BSA places the full resolving power of the analyzer into any selected band of the measurement, thus increasing the resolution and providing a more detailed picture. Another important benefit of this increased resolution is the reduction in the average noise power level which makes signal detection easier. BSA is a vital aid in improving power spectrum and transfer function measurements.*
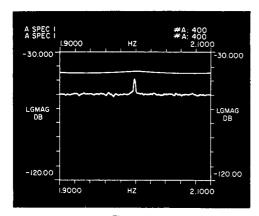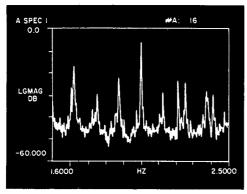
Figure 9

**Fig. 9** *A BSA measurement was made on the same signal as in Figure 7. The upper trace is the original auto power measurement from Figure 8 repeated for reference. The BSA bandwidth is from 1.9 to 2.1 orders. This improvement in resolution (32 to 1) reduces the overall noise level and clearly shows the second order.*



Figure 10

**Fig. 10** *Another BSA measurement was made on another machine. In this case, the machine exhibited a great deal of bearing noise and other shaft-related vibrations. These are clearly visible using the auto power spectrum without the need of a trigger signal.*
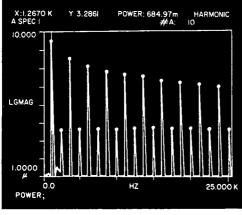


Figure 11

**Fig. 11** *A typical auto spectrum of a square wave. The intensified dots indicate harmonically related frequencies of the fundamental at 1267 Hz. Note the predominence of odd-order harmonics characteristic of a square wave. By ratioing the power in the harmonics (0.685V²) to that in the fundamental (3.29V²), the total harmonic distortion can be calculated.*

## CROSS POWER SPECTRUM

The cross power spectrum is a measure of the mutual power between two signals at each frequency in the analysis band. It is defined as:

$$G_{yx}(f) = S_y(f) \cdot S_x(f)^*$$

and is the dual of the cross correlation function.

The cross spectrum $G_{yx}(f)$ contains both magnitude and phase information. The phase of $G_{yx}(f)$ at each frequency is the relative phase between the two signals. As such, $G_{yx}(f)$ can be averaged without the need of a qualifying trigger.

The magnitude of $G_{yx}(f)$ is the product of the magnitudes of the two signals. Where both signals are high, the cross product will be high; where they are both low, the product will be low. This makes $G_{yx}(f)$ an extremely sensitive tool for isolating major signals that are common to both input and output. However, keep in mind when analyzing a system that there may be signals in the output not necessarily caused by the input which will still show up in the cross spectrum. In other words, the cross spectrum does not necessarily give causal relationships.

The cross spectrum is mainly used to analyze phase relationships between signals caused, for example, by time delays in a system, propagation delays, or multiple signal paths between source and destination.
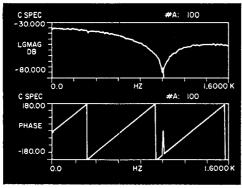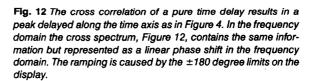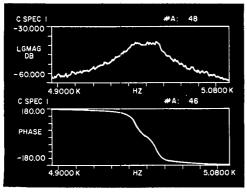
Figure 12

**Fig. 12** *The cross correlation of a pure time delay results in a peak delayed along the time axis as in Figure 4. In the frequency domain the cross spectrum, Figure 12, contains the same information but represented as a linear phase shift in the frequency domain. The ramping is caused by the ±180 degree limits on the display.*

Figure 13

**Fig. 13** *A cross spectrum measurement of two closely coupled modes of a mechanical system. For each complex mode, there is a 180 degree phase shift. Although it is not definitely clear from the magnitude that two resonances exist, the phase plot shows it clearly.*

## TRANSFER FUNCTION

The transfer function is the mathematical description of the input-output relationship of a system. A system can be an electronic filter, an automobile engine, a vibrating airplane wing, or an organ of the human body. For single inputs and outputs, which is usually the case, the transfer function can be computed as:

$$H(f) = \frac{\overline{G_{yx}(f)}}{\overline{G_{xx}(f)}}$$

The transfer function gives both magnitude and phase information but, unlike the cross power spectrum, its magnitude is normalized by $G_{xx}(f)$. And, since the measured phase is that of the cross spectrum, $G_{yx}(f)$, a trigger signal (time synchronization) is not required.

Depending on the application, transfer function measurements are known by a variety of names:

- Frequency Response
- Bode Plot
- Dynamic Mass
- Mechanical Impedance
- Dynamic Stiffness

- Compliance
- Mobility
- Inertance
- System Response
- Gain

Transfer function analysis is used, for example, to characterize closed-loop control systems, quiet noisy automobile chassis, eliminate flutter from aircraft wings, test loudspeaker designs, and monitor nuclear reactor core movement.
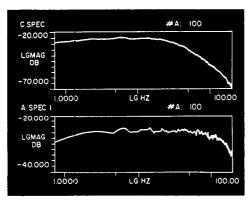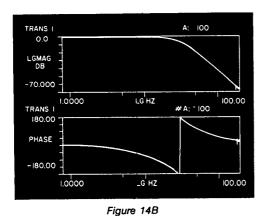


Figure 14A

**Fig. 14A** *The cross spectrum and input auto spectrum are the basis for computing the transfer function.*

Figure 14B

**Fig. 14B** *The transfer function measurement shown is computed from these functions. It is clear that normalizing by $G_{xx}(f)$ removes the deterministic error in the measurement by taking into account the actual shape of the drive spectrum $G_{xx}(f)$. In addition, the first-order effects of randomness are also removed since the sample records of x(t) and y(t) are related by a linear system and have essentially the same random variation.*
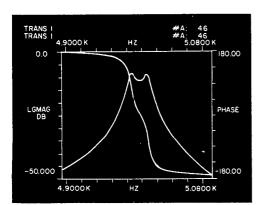


Figure 15

**Fig. 15** *The transfer function magnitude and phase of two closely coupled modes are superimposed. This is the same system as measured in Figure 13. Note that the phase of the transfer function and cross spectrum are identical.*

## COHERENCE FUNCTION

The coherence function is a measure of the degree of causality between system input and output. It is defined as:

$$\gamma^2(f) = \frac{\overline{G_{yx}(f) \cdot G_{yx}(f)^*}}{\overline{G_{xx}(f) \cdot G_{yy}(f)}} \qquad 0 \leqslant \gamma^2 \leqslant 1$$

$\gamma^2(f)$ is a measure of the signal strength in the output, y, that is due to the measured input, x, at each frequency. $\gamma^2(f)=1$ indicates that all of the output signal at that frequency is due to the measured input. If $\gamma^2(f)<1$, then there are extraneous input signals not being measured, noise present in the system, non-linearities in the system, or time delays through the system.

Another way of looking at the coherence function is as a measure of the signal-to-noise ratio (S/N) of the measurement. The equation is:

$$\frac{S(f)}{N(f)} = \frac{\gamma^2(f)}{1-\gamma^2(f)}$$

Signal-to-noise ratio is frequently a more useful term, since its meaning is easier to visualize. Both the coherence function and signal-to-noise ratio are used in conjunction with the transfer function as an indication of the quality of the measurement. A small value of $\gamma^2(f)$ does not necessarily mean that the transfer function measured at that frequency is invalid. It may only mean that a great deal of averaging must be done to improve the signal-to-noise ratio.

The coherence function can also be used to 'weight' the output power spectrum $G_{yy}(f)$ in at least two useful ways. Multiplying $G_{yy}(f)$ by $\gamma^2(f)$ gives the coherent output power $G_{yy}(f) \cdot \gamma^2(f)$, which is the power at the output due to the input. This is useful in isolating the contributions of multiple sources. In a similar fashion, $G_{yy}(f)(1-\gamma^2(f))$ is the power at the output due to system noise.
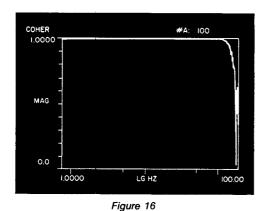


Figure 16

**Fig. 16** *The coherence function for the measurement shown in Figure 15. The signal-to-noise ratio here is infinite over most of the range, and the noise signal contribution to the measurement is zero.*
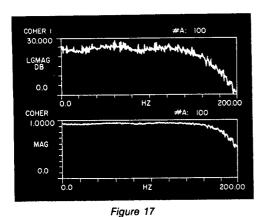
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Figure 17

**Fig. 17** When the same measurement is repeated in the presence of extraneous noise the coherence function is less than one. The bottom trace is the coherence function and is slightly less than one. The top trace shows the signal-to-noise ratio calculated from this coherence.
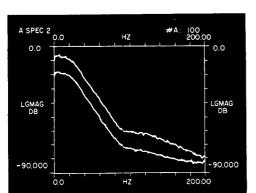


Figure 18

**Fig. 18** The upper trace shows the output power spectrum, $G_{yy}(f)$, the lower trace $G_{yy}(f) \cdot (1-\gamma^2(f))$. From Figure 18 we can determine that the noise contribution is around 10 to 13 dB down from the total. This corresponds to the 5% to 10% noise (90% to 95% coherence) shown in Figure 17.

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